Saint-Venant's principle in dynamical porous thermoelastic media with memory for heat flux

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Abstract – In the present paper, we study a linear thermoelastic porous material with a constitutive equation for heat flux with memory. An approximated theory of thermodynamics is presented for this model and a maximal pseudo free energy is determined. We use this energy to study the spatial behaviour of the thermodynamic processes in porous materials. We obtain the domain of influence theorem and establish the spatial decay estimates inside of the domain of influence. Further, we prove a uniqueness theorem valid for finite or infinite body. The body is free of any kind of a priori assumptions concerning the behaviour of solutions at infinity.

1 Introduction

The Saint–Venant's principle has a central role in more theoretical and applied questions of elasticity. An important review of research on the spatial behaviour of solutions for statical and dynamical problems was given by Horgan and Knowles [1] and Horgan [2, 3]. Relevant information on the spatial behaviour of the solutions for the dynamical problems of elasticity are given by the domain of influence theorem as it is presented by Gurtin [4]. A further study in this connection was made by Chiriţă [5] for the linear theory of thermoelasticity, where a bounded solid is subjected to the action of nonzero boundary loads only on the end face.

Recently, Fabrizio, Lazzari and Munoz Rivera [6] have studied a linear thermoelastic material which exhibits a constitutive equation for heat flux with memory. Some existence, uniqueness and asymptotic theorems have been established in connection with this approach. Further, Chiriţă and Lazzari [7] have studied the spatial behaviour of the thermoelastic processes. The hereditary effects are taken into account in a dissipative boundary condition by Ciarletta [8] and Bartilomo and Passarella [9].

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On the other hand, Ieşan [10] has developed a linear theory of thermoelastic materials with voids, by generalizing some ideas of Cowin and Nunziato [11]. In this theory the bulk density is written as the product of two fields, the matrix material density field and the volume fraction field. This representation introduces an additional degree of kinematic freedom and it is compatible with the theory of granular materials developed by Goodman and Cowin [12]. The spatial and temporal behaviour in linear thermoelasticity of materials with voids has been studied by Chiriţă and Scalia in [13]. Further, it is shown by Iovane and Passsarella in [14] the spatial exponential decay for the boundary–final value problem associated to elastic porous materials with a memory effect for the intrinsic equilibrated body forces.

In the present paper, we study a linear thermoelastic porous material which exhibits effects of fading memory in the constitutive equation for the heat flux. Such model exhibits finite speeds for the propagation of thermal disturbances and hence leads to accurate modeling of the transient thermal behaviour. By applying the thermodynamical laws, we can prove the existence of a pseudofree energy potential such that the internal dissipation vanishes despite of the dissipative character of the model. Then, we use this energy for studying the spatial behaviour of thermodynamical processes.

The paper is organized as follows. In Section 2, we present the thermodynamical restrictions imposed by the thermodynamical laws on the model in question; then, we introduce a maximal pseudofree energy. In Section 3, we deduce a result for describing the domain of influence by using an appropriate time-weighted surface measure. This domain at instant $t \in [0,\ T]$ is the set $D_{\zeta t}$ of all points of continuum for which $r \leq \zeta t$, where r represents the distance from the bounded support $\hat{D}_T$ of the (external) given data in the time interval $[0,\ T]$ and ζ is a constant characteristic to the thermodynamic coefficients. We obtain, into the inner of the domain of influence, a spatial decay of exponential type. The decay rate, characterized by a factor independent of time, is $\exp(-\dfrac{\sigma}{\zeta}r)$, where σ is a positive parameter independent of time and defining the time-weighted surface power measure. In Section 4, we prove that, into the inner of the domain of influence $D_{\zeta t}$, an energetic measure associated with the thermodynamic process tends to zero with a decay rate equal to $\left(1 - \frac{r}{\zeta t}\right)$.

2 Basic equations. Thermodynamic restrictions

We consider a porous thermoelastic body that occupies a (regular) region B of the physical space $\mathbb{R}^3$ in an assigned reference configuration. By identifying $\mathbb{R}^3$ with the associated vector space, an orthonormal system of reference is introduced, so that vectors and tensors will have components denoted by Latin subscripts ranging over 1, 2, 3. The letters in boldface stand for the tensor $\mathbf{L}$ of an order $p \geq 1$, and $L_{ij\,.\,.\,.\,k}$ (p subscripts) are the components of the tensor $\mathbf{L}$. Summation over repeated subscripts and other typical conventions for differential operations are implied, such as a superposed dot or a comma followed by a subscript to denote partial derivative with respect to time or with respect to the corresponding coordinate.

We denote $\mathbf{U} \equiv \{\mathbf{u},\ \varphi,\ \theta\}$, where $\mathbf{u}$ is the displacement vector fields, φ is the change in volume fraction starting from the reference configuration, θ is the temperature variation from the uniform reference temperature $\theta_0(>0)$. Further, ρ is the bulk mass density and χ is the equilibrated inertia in the reference state.

We restrict our attention to the linear theory of thermoelasticity in which the effect of fading memory is contained in the thermal gradient $\nabla\theta$. The local balance equations become [10] for the present problem

$$\begin{aligned} & S_{ji,j} + \rho f_i = \rho \ddot{u}_i, \\ & h_{j,j} + g + \rho\ell = \rho\chi\ddot{\varphi}, \\ & -q_{j,j} + \rho r = \rho\tau, \qquad\qquad \text{on } B \times (0,\infty). \end{aligned} \tag{1}$$

In the previous equations:

$\mathbf{S}$ and $\mathbf{f}$ are the stress tensor and body force, respectively;
$\mathbf{h}$, g and ℓ are the equilibrated stress vector, intrinsic and extrinsic equilibrated body force, respectively;
τ, $\mathbf{q}$ and r are the rate at which heat is absorbed for a unit of volume, the heat flux vector and the (extrinsic) heat supply, respectively.

Let $\mathbf{E}$ be the strain field associated with $\mathbf{u}$, $\bigtriangledown\theta^t$ be the history up to time t for the thermal gradient and $\overline{\bigtriangledown\theta}^t$ be the integrated history of the thermal gradient, i. e.

$$E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2}$$

$$\theta^t_{,j}(s) = \theta_{,j}(t-s), \qquad \bar{\theta}^t_{,j}(s) = \int_{t-s}^{t} \theta_{,j}(\alpha)d\alpha, \qquad \forall s \in [0,+\infty). \tag{3}$$

As it was shown by Ieşan in [10], the constitutive equations are

$$\begin{aligned} & S_{ij} = C_{ijrs}E_{rs} + D_{ijs}\varphi_{,s} + B_{ij}\varphi + M_{ij}\theta, \\ & h_i = D_{rsi}E_{rs} + A_{is}\varphi_{,s} + b_i\varphi + a_i\theta, \\ & g = -B_{rs}E_{rs} - b_s\varphi_{,s} - \xi\varphi - m\theta, \\ & \rho\tau = -\theta_0 M_{rs}\dot{E}_{rs} - \theta_0 a_s\dot{\varphi}_{,s} - \theta_0 m\dot{\varphi} + \rho c\dot{\theta}, \end{aligned} \tag{4}$$

where the material coefficients of eqs. (4) are supposed continuous and bounded fields on $\bar{B}$ (the set $\bar{B}$ is the closure of B). Moreover, they satisfy the symmetry relations

$$\begin{aligned} & C_{ijrs} = C_{rsij} = C_{jirs}, \quad D_{ijr} = D_{jir}, \quad A_{ij} = A_{ji}, \\ & B_{ij} = B_{ji}, \quad M_{ij} = M_{ji}. \end{aligned} \tag{5}$$

Further, we consider for the heat flux the following constitutive equation as suggested by Fabrizio, Lazzari and Munoz Rivera in [6]

$$q_i(t) = -\theta_0 \int_0^\infty K_{ij}(s)\theta^t_{,j}(s)ds, \tag{6}$$

with the relaxation thermal conductivity $\mathbf{K}$. We are assuming K_{ij} are continuous and bounded fields on $\bar{B}$; moreover, $K_{ij}(s) \in L^2(0,\infty)$ and they satisfy the relations

$$K_{ij}(s) = K_{ji}(s), \qquad K_{ij}(\infty) = 0. \tag{7}$$

By taking into account $\bar{\theta}^t_{,j}(0) = 0$ and

$$\frac{d}{dt}\bar{\theta}^t_{,j}(s) = \theta_{,j}(t) - \theta^t_{,j}(s), \qquad \frac{d}{ds}\bar{\theta}^t_{,j}(s) = \theta^t_{,j}(s), \tag{8}$$

the constitutive equation (6) can be written as

$$q_i(t) = \theta_0 \int_0^\infty \dot{K}_{ij}(s)\bar{\theta}^t_{,j}(s)ds. \tag{9}$$

In what follows, we study the restriction imposed by the fundamental laws of thermodynamics in terms of cyclic processes. With this aim, we recall the laws of the thermodynamics as considered in [6, 15, 16]:

First Law of Thermodynamics: *For every cyclic process the following equality holds*

$$\oint \left\{\rho\tau(t) + S_{rs}(t)\dot{E}_{rs}(t) + h_s(t)\dot{\varphi}_{,s}(t) - g(t)\dot{\varphi}(t)\right\} dt = 0. \tag{10}$$

Second Law of Thermodynamics: *For every cyclic process the following equality holds*

$$\frac{1}{\theta_0^2}\oint \left\{\rho\tau(t)(\theta_0 - \theta(t)) + q_s(t)\theta_{,s}(t)\right\} dt \leq 0, \tag{11}$$

and the equality sign holds if and only if the process is reversible.

The relations (6, 11) imply

$$\int_0^d \left(\int_0^\infty K_{ij}(s)\theta^t_{,j}(s)ds\right)\theta_{,i}(t)dt \geq 0 \tag{12}$$

for any cycle on $[0,\ d)$, and the equality sign holds if and only if $\bigtriangledown\theta^t$ is a constant history.

If we put $\theta_{,i}(t) = \bar{k}_i\cos(\omega t) + \tilde{k}_i\sin(\omega t)$ into the relation (12) with $d = \frac{2\pi}{\omega}$, then we obtain

$$\int_0^\infty (\bar{k}_i K_{ij}(s)\bar{k}_j + \tilde{k}_i K_{ij}(s)\tilde{k}_j)\cos(\omega s)ds + \int_0^\infty (\tilde{k}_i K_{ij}(s)\bar{k}_j - \bar{k}_i K_{ij}(s)\tilde{k}_j)\sin(\omega s)ds \geq 0, \tag{13}$$

where ω is strictly positive and $\bar{k}_i$, $\tilde{k}_i$ are the continuous functions on $\bar{B}$.

Now, we introduce the Fourier transform, sine and cosine transforms of function $f \in L^2(-\infty,\ \infty)$, i. e.

$$\begin{aligned} f^F(\omega) &= \int_{-\infty}^{\infty} f(\xi)\exp(-i\omega\xi)d\xi, \\ f^S(\omega) &= \int_0^{\infty} f(\xi)\sin(\omega\xi)d\xi, \quad f^C(\omega) = \int_0^{\infty} f(\xi)\cos(\omega\xi)d\xi. \end{aligned} \tag{14}$$

We remark that the Fourier inversion formula imply

$$f(\xi) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f^S(\xi)\sin(\omega\xi)d\omega; \tag{15}$$

and the Plancherel's theorem of the Fourier transform gives

$$\int_{-\infty}^{\infty} f(\xi)g(\xi)d\xi = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} f^F(\xi)g^{F\,*}(\xi)d\omega, \tag{16}$$

with $f,\ g \in L^2$(-$\infty,\ \infty$) and with g_F^* is the complex conjugate of g_F.

We can easily prove by eqs. (8, 14) that

$$\dot{\bar{\theta}}^{tC}_{,i} = -\omega\bar{\theta}^{tS}_{,i}, \qquad\qquad \dot{\bar{\theta}}^{tS}_{,i} = \frac{1}{\omega}\theta_{,i}(t) + \omega\bar{\theta}^{tC}_{,i}. \tag{17}$$

By putting $\bar{k}_i = \tilde{k}_i$=k_i in the relation (13) as Chirita and Lazzari in [7], we obtain

$$k_i K^C_{ij} k_j > 0, \qquad\qquad \forall\mathbf{k} \equiv (k_1, k_2, k_3) \neq \mathbf{0}, \qquad \forall\omega > 0. \tag{18}$$

Thus, the relations (17, 18) imply that $\dot{\mathbf{K}}^S$ is a negative definite tensor.

By using eqs. (9, 15, 16), we have

$$q_i(t) = \sqrt{\frac{2}{\pi}}\theta_0\int_0^{\infty} \dot{K}^S_{ij}(\omega)\bar{\theta}^{t\,S}_{,j}(\omega)d\omega, \tag{19}$$

and

$$\dot{K}_{ij}(\xi) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \dot{K}^S_{ij}(\omega)\sin(\xi\omega)d\omega. \tag{20}$$

If we integrate eq. (20) with the respect ξ and we take into account the Riemann Lebesgue lemma and the hypotheses $K_{ij}(\infty) = 0$, then it follows

$$K_{ij}(0) = -\sqrt{\frac{2}{\pi}}\int_0^{\infty} \frac{1}{\omega}\dot{K}^S_{ij}(\omega)d\omega. \tag{21}$$

The relations (7, 21) and the negative definiteness of the tensor $\dot{\mathbf{K}}^S$ imply that $\mathbf{K}(0)$ is a symmetric and positive definite tensor, i.e.

$$k_i K_{ij}(0) k_j > 0, \qquad \forall \mathbf{k} \equiv (k_1, k_2, k_3) \neq \mathbf{0}, \qquad \forall \omega > 0. \tag{22}$$

Consequently, it follows that

$$K_{ij}(0) k_i k_j \leq K_M k_i k_i, \qquad \forall \mathbf{k} \; = \{k_1, k_2, k_3\}, \tag{23}$$

where $K_M(\mathbf{x}) > 0$ is the largest characteristic eigenvalue of $\mathbf{K}(\mathbf{x},\, 0)$.

In what follows, we will assume that the bulk mass density ρ, the equilibrated inertia χ and the constant heat c are strictly positive, continuous and bounded fields on $\bar{B}$, so that

$$0 < \; \rho_0 \equiv \inf_{\mathbf{x} \in \bar{B}} \rho, \qquad 0 < \chi_0 \equiv \inf_{\mathbf{x} \in \bar{B}} \chi, \qquad 0 < c_0 \equiv \inf_{\mathbf{x} \in \bar{B}} c. \tag{24}$$

Moreover, we suppose that $\mathbf{K}(\mathbf{x},\, 0)$ is continuous and bounded on $\bar{B}$. By setting

$$K_0 \equiv \sup_{\mathbf{x} \in \bar{B}} \frac{\theta_0 K_M}{c}, \tag{25}$$

we obtain

$$K_{ij}(\mathbf{x},\, 0) k_i k_j \leq K_0 \frac{c}{\theta_0} k_i k_i \qquad \forall \mathbf{k} \; = \{k_1, k_2, k_3\} \neq \mathbf{0}. \tag{26}$$

Now, we introduce the vector space $\mathcal{V}$ of all vector fields of the form

$$\tilde{\mathbf{F}} \equiv \{F_{ij}, \sqrt{\chi}\pi_i, \psi\}, \qquad \text{with} \qquad F_{ij} = F_{ji}.$$

For any $\tilde{\mathbf{F}}, \bar{\mathbf{F}} \in \mathcal{V}$, we define the inner product and the magnitude by

$$\tilde{\mathbf{F}} \cdot \bar{\mathbf{F}} = F_{ij}\bar{F}_{ij} + \chi \pi_i \bar{\pi}_i + \psi \bar{\psi}, \qquad \left|\tilde{\mathbf{F}}\right|^2 = F_{ij}F_{ij} + \chi \pi_i \pi_i + \psi^2, \tag{27}$$

where $\bar{\mathbf{F}} \equiv \{\bar{F}_{ij}, \sqrt{\chi}\bar{\pi}_i, \bar{\psi}\}$. For given $\tilde{\mathbf{F}} \in \mathcal{V}$, we introduce the vector field $\tilde{\mathbf{S}}(\tilde{\mathbf{F}}) \in \mathcal{V}$ as

$$\tilde{\mathbf{S}}(\tilde{\mathbf{F}}) \equiv \left\{ \tilde{S}_{ij}(\tilde{\mathbf{F}}), \sqrt{\chi}\left(\frac{1}{\chi}\tilde{h}_i(\tilde{\mathbf{F}})\right), -\tilde{g}(\tilde{\mathbf{F}}) \right\}, \tag{28}$$

where

$$\begin{aligned} \tilde{S}_{ij}(\tilde{\mathbf{F}}) &= C_{ijrs}F_{rs} + D_{ijs}\pi_s + B_{ij}\psi, \\ \tilde{h}_i(\tilde{\mathbf{F}}) &= D_{rsi}F_{rs} + A_{ij}\pi_j + b_i\psi, \\ \tilde{g}(\tilde{\mathbf{F}}) &= -B_{ij}F_{ij} - b_i\pi_i - \xi\psi, \end{aligned} \tag{29}$$

and the coefficients are defined in eqs. (4). They obey to the symmetry relations (5).

As in [13, 14], we consider the following bilinear form

$$
\begin{aligned}
2\mathcal{F}(\tilde{\mathbf{F}}, \bar{\mathbf{F}}) = C_{ijrs}F_{ij}\bar{F}_{rs} + \xi\psi\bar{\psi} + A_{ij}\pi_i\bar{\pi}_j + B_{ij}(F_{ij}\bar{\psi} + \bar{F}_{ij}\psi) + \\
+ D_{ijs}(F_{ij}\bar{\pi}_s + \bar{F}_{ij}\pi_s) + b_i(\psi\bar{\pi}_i + \bar{\psi}\pi_i).
\end{aligned}
\tag{30}
$$

From (29, 30) we have

$$
2\mathcal{F}(\tilde{\mathbf{F}}, \bar{\mathbf{F}}) = \tilde{S}_{ij}(\tilde{\mathbf{F}})\bar{F}_{ij} + \tilde{h}_i(\tilde{\mathbf{F}})\bar{\pi}_i - \tilde{g}(\tilde{\mathbf{F}})\bar{\psi}. \tag{31}
$$

Throughout this paper, we will assume that the quadratic form $\tilde{W}$ associated to $\mathcal{F}$ is positive definite, so that

$$
2\tilde{W}(\tilde{\mathbf{F}}) = 2\mathcal{F}(\tilde{\mathbf{F}}, \tilde{\mathbf{F}}) \leq \mu_M \left(F_{ij}F_{ij} + \chi\ \pi_i\pi_i + \psi^2\right), \tag{32}
$$

where $\mu_M > 0$. In particular, in the case $\tilde{\mathbf{E}} \equiv \{E_{ij},\ \sqrt{\chi}\varphi_{,i},\ \varphi\}$, from (4, 29) we get

$$
S_{ij} = \tilde{S}_{ij}(\tilde{\mathbf{E}}) + M_{ij}\theta, \qquad h_i = \tilde{h}_i(\tilde{\mathbf{E}}) + a_i\theta, \qquad g = \tilde{g}(\tilde{\mathbf{E}}) - m\theta. \tag{33}
$$

Taking into account eqs. (29, 30, 32, 33), we deduce

$$
\begin{aligned}
W^* &\equiv \tilde{W}(\tilde{\mathbf{E}}) = \tfrac{1}{2}[(S_{ij} - M_{ij}\theta)E_{ij} + (h_i - a_i\theta)\varphi_{,i} - (\ g + m\theta)\varphi], \\
\dot{W}^* &= S_{ij}\dot{E}_{ij} + h_i\dot{\varphi}_{,i} -\ g\dot{\varphi} + \frac{\rho\tau\theta}{\theta_0} - \frac{\rho\theta}{\theta_0}\dot{\theta},
\end{aligned}
\tag{34}
$$

and

$$
2W^* \leq \mu_M \left(E_{ij}E_{ij} + \chi\ \varphi_{,i}\varphi_{,i} + \varphi^2\right). \tag{35}
$$

By setting $\bar{\mathbf{F}} = \tilde{\mathbf{S}}(\tilde{\mathbf{E}})$ in the relations (31,32), we have

$$
2\tilde{W}(\tilde{\mathbf{S}}(\tilde{\mathbf{E}})) \leq \mu_M \left[\tilde{S}_{ij}(\tilde{\mathbf{E}})\tilde{S}_{ij}(\tilde{\mathbf{E}}) + \frac{1}{\chi}\tilde{h}_i(\tilde{\mathbf{E}})\tilde{h}_i(\tilde{\mathbf{E}}) + \tilde{g}^2(\tilde{\mathbf{E}})\right]. \tag{36}
$$

Therefore, we conclude thanks to (31, 36) and thanks to Cauchy-Schwarz's inequality that

$$
\begin{aligned}
\left[\tilde{S}_{ji}(\tilde{\mathbf{E}})\tilde{S}_{ji}(\tilde{\mathbf{E}}) + \frac{1}{\chi}\tilde{h}_i(\tilde{\mathbf{E}})\tilde{h}_i(\tilde{\mathbf{E}}) + \tilde{g}^2(\tilde{\mathbf{E}})\right]^2 = \left[2\mathcal{F}(\tilde{\mathbf{E}}, \tilde{\mathbf{S}}(\tilde{\mathbf{E}}))\right]^2 \leq 4W^*\tilde{W}(\tilde{\mathbf{S}}(\tilde{\mathbf{E}})) \leq \\
\leq 2W^*\mu_M \left[\tilde{S}_{ji}(\tilde{\mathbf{E}})\tilde{S}_{ji}(\tilde{\mathbf{E}}) + \frac{1}{\chi}\tilde{h}_i(\tilde{\mathbf{E}})\tilde{h}_i(\tilde{\mathbf{E}}) + \tilde{g}^2(\tilde{\mathbf{E}})\right].
\end{aligned}
$$

Consequently, it follows

$$
\left|\tilde{\mathbf{S}}(\tilde{\mathbf{E}})\right|^{\mathbf{2}} = \tilde{S}_{ij}(\tilde{\mathbf{E}})\tilde{S}_{ij}(\tilde{\mathbf{E}}) + \frac{1}{\chi}\tilde{h}_i(\tilde{\mathbf{E}})\tilde{h}_i(\tilde{\mathbf{E}}) + \tilde{g}^2(\tilde{\mathbf{E}}) \leq 2\mu_M W^*. \tag{37}
$$

Thanks to the inequality for second-order tensors $\mathbf{L}$ and $\mathbf{G}$

$$
(L_{ij} + G_{ij})(L_{ij} + G_{ij}) \leq (1+\epsilon)L_{ij}L_{ij} + (1 + \frac{1}{\epsilon})G_{ij}G_{ij}, \qquad \forall\epsilon > 0,
$$

and thanks to eqs. (33, 37), we get

$$
\begin{aligned}
S_{ij}S_{ij}+\frac{1}{\chi}h_ih_i+g^2 &\leq (1+\epsilon)\left|\tilde{\mathbf{S}}(\mathbf{E})\right|^{\mathbf{2}}+(1+\frac{1}{\epsilon})\left\{M_{ij}M_{ij}+\frac{1}{\chi}a_ia_i+m^2\right\}\theta^2\leq \\
&\leq (1+\epsilon)2\mu W^*+(1+\frac{1}{\epsilon})M\frac{\rho c}{\theta_0}\theta^2, \qquad \forall\epsilon>0,
\end{aligned} \tag{38}
$$

with

$$
\mu\equiv\sup_{\mathbf{x}\in\bar{B}}\mu_M, \qquad M\equiv\sup_{\mathbf{x}\in\bar{B}}\frac{\theta_0}{\rho c}\left(M_{ij}M_{ij}+\frac{1}{\chi}a_ia_i+m^2\right). \tag{39}
$$

The Laws of thermodynamics imply the existence of the thermodynamical internal potential energy e and the entropy η such that

$$
\rho\dot{e}(t) = \rho\tau(t)+S_{ji}(t)\dot{E}_{ji}(t)+h_j(t)\dot{\varphi}_{,j}(t)-g(t)\dot{\varphi}(t), \tag{40}
$$

$$
\rho\dot{\eta}(t) \geq \frac{1}{\theta_0^2}\left\{\rho\tau(t)(\theta_0-\theta(t)+q_j(t)\theta_{,j}(t)\right\}. \tag{41}
$$

We introduce the pseudofree energy potential

$$
\rho\Psi(t)=\rho e(t)-\rho\theta_0\eta(t). \tag{42}
$$

Moreover, we define the maximal pseudofree potential energy Ψ_M such that

$$
\rho\dot{\Psi}_M(t)=\frac{\rho\tau(t)\theta(t)}{\theta_0}+S_{ji}(t)\dot{E}_{ji}(t)+h_j(t)\dot{\varphi}_{,j}(t)-g(t)\dot{\varphi}(t)-\frac{q_j(t)\theta_{,j}(t)}{\theta_0}. \tag{43}
$$

Then, the relations (40-43) imply

$$
\rho\dot{\Psi}(t)\leq\rho\dot{\Psi}_M(t). \tag{44}
$$

By taking into account the relations (4, 19, 34_2), a maximal pseudofree energy potential is

$$
\begin{aligned}
\rho\Psi_M(t)=W^*(t)+\frac{\rho c\theta^2(t)}{2\theta_0}-\frac{1}{\pi}\int_0^\infty\omega\{&\dot{K}^S_{ij}(\mathbf{x},\omega)\bar{\theta}^{tS}_{,i}(\mathbf{x},\omega)\bar{\theta}^{tS}_{,j}(\mathbf{x},\omega)+ \\
+&\dot{K}^S_{ij}(\mathbf{x},\omega)\bar{\theta}^{tC}_{,i}(\mathbf{x},\omega)\bar{\theta}^{tC}_{,j}(\mathbf{x},\omega)\}d\omega.
\end{aligned} \tag{45}
$$

Since $\tilde{W}$ is a positive definite quadratic form, ρ, χ and c are strictly positive, $\dot{\mathbf{K}}^S$ is the negative definite tensor, we deduce that the functional Ψ_M is a norm. As in [7], we can prove by eq. (19) that

$$
\begin{aligned}
|\mathbf{q}(t)|^2 &= \frac{2\theta_0}{\pi}\int_0^\infty q_i(t)\dot{K}^S_{ij}(\mathbf{x},\omega)\bar{\theta}^{tS}_{,j}(\mathbf{x},\omega)d\omega\leq && (46)\\
&\leq \theta_0\left(-\frac{2}{\pi}\int_0^\infty\frac{1}{\omega}q_i(t)q_j(t)\dot{K}^S_{ij}(\omega)d\omega\right)^{1/2}\left(-\frac{2}{\pi}\int_0^\infty\omega\dot{K}^S_{ij}(\omega)\bar{\theta}^{tC}_{,i}(\omega)\bar{\theta}^{tS}_{,j}(\omega)d\omega\right)^{1/2}.
\end{aligned}
$$

The relations (21, 25, 26, 45, 46) imply

$$
|\mathbf{q}(t)|^2\leq K_0\theta_0c\left\{2\rho\Psi_M(t)-2W^*(t)-\frac{\rho c\theta^2(t)}{\theta_0}\right\}. \tag{47}
$$

3 A time-weighted surface power measure

Throughout this work by an admissible process we mean an ordered array $\pi \equiv [\mathbf{u}, \mathbf{E}, \mathbf{S}, \varphi, \gamma, \mathbf{h}, \theta, \mathbf{k}, \mathbf{q}]$ with the following properties

i. $u_i, \varphi \in C^{2,2}(\bar{B} \times [0, +\infty)), \quad \theta \in C^{1,1}(\bar{B} \times [0, +\infty))$;

ii. $E_{ij} = E_{ji}, \quad \gamma_i = \varphi_{,i}, \quad k_i = \bar{\theta}^t_{,i} \in C^{0,1}(\bar{B} \times [0, +\infty))$;

iii. $S_{ij} = S_{ji}, \quad h_i, q_i \in C^{1,0}(\bar{B} \times [0, +\infty))$,

and which meets the equations of motion (1), the geometrical equations (2, 3), the constitutive equations (4, 6) and the following initial conditions

$$u_i(0) = u_i^0, \qquad \dot{u}_i(0) = \dot{u}_i^0, \qquad \varphi(0) = \varphi^0, \qquad \dot{\varphi}(0) = \dot{\varphi}^0, \qquad \theta(0) = \vartheta_0. \tag{48}$$

Let $\mathbf{s}$, h and q be the surface tractions, the surface equilibrated stress and the heat flux, so

$$s_i(t) = S_{ji}(t) n_j, \qquad h(t) = h_j(t) n_j, \qquad q(t) = q_j(t) n_j. \tag{49}$$

We denote by $\Gamma \equiv [f_i, s_i, u_i^0, \dot{u}_i^0, l, h, \varphi^0, \dot{\varphi}^0, r, q, \vartheta_0, \theta^0_{,j}]$ the external data and assume that all functions are prescribed continuous functions.

Now, we introduce the support $\hat{D}_T$ of the external data Γ and the body supplies on the time interval $[0, T]$, i. e. the set of all $\mathbf{x} \in \overline{B}$ such that:

(1) if $\mathbf{x} \in B$, then

$$u_i^0 \neq 0 \text{ or } \dot{u}_i^0 \neq 0 \text{ or } \varphi^0 \neq 0 \text{ or } \dot{\varphi}^0 \neq 0 \text{ or } \vartheta_0 \neq 0$$

or

$$\theta^0_{,i}(s) \neq 0 \;\text{ for some } s \in (-\infty, 0] \tag{50}$$

or

$$f_i(s) \neq 0 \;\text{ or }\; \ell(s) \neq 0 \;\text{ or }\; r(s) \neq 0 \;\text{ for some } s \in [0, T];$$

(2) if $\mathbf{x} \in \partial B$, then

$$s_i(s)\dot{u}_i(s) \neq 0 \text{ or } h(s)\dot{\varphi}(s) \neq 0 \text{ or } q(s)\theta(s) \neq 0 \text{ for some } s \in [0, T].$$

In what follows, we will assume that $\hat{D}_T$ is a bounded set.

We introduce a nonempty set $\hat{D}_T^*$ such that $\hat{D}_T \subset \hat{D}_T^* \subset \bar{B}$ and

(a) if $\hat{D}_T \cap B \neq \emptyset$, then we choose $\hat{D}_T^*$ to be the smallest bounded regular region in $\bar{B}$ that includes $\hat{D}_T$; in particular, we set $\hat{D}_T^* = \hat{D}_T$ if $\hat{D}_T$ it is a regular region;

(b) if $\emptyset \neq \hat{D}_T \subset \partial B$, then we choose $\hat{D}_T^*$ to be the smallest regular subsurface of ∂B that includes $\hat{D}_T$; in particular, we set $\hat{D}_T^* = \hat{D}_T$ if $\hat{D}_T$ is a regular subsurface of ∂B;

(c) if $\hat{D}_T = \emptyset$, then we choose $\hat{D}_T^*$ to be an arbitrary nonempty regular subsurface of ∂B.

On this basis, we introduce the set D_r, by

$$D_r = \{\mathbf{x} \in \overline{B} : \hat{D}_T^* \cap \overline{\Sigma(r)} \neq \emptyset\},$$

where $\Sigma(r)$ is the open ball with radius r and center at $\mathbf{x}$.

Further, we shall use the notation B_r for the part of B contained in $B \setminus D_r$ and we set $B(r_1,\ r_2) = B_{r_2} \setminus B_{r_1},\ r_1 \geq r_2$. We denote by S_r the subsurface of ∂B_r contained into the inner of B and whose outward unit normal vector is forwarded to the exterior of D_r. We can observe that the data are null on $B_r,\ S_r$.

We define the following time-weighted surface power function $I(r,\ t)$

$$I(r,t) = -\int_0^t \int_{S_r} e^{-\sigma s}[s_i(s)\dot{u}_i(s) + h(s)\dot{\varphi}(s) - \frac{1}{\theta_0} q(s)\theta(s)]da ds. \tag{51}$$

for a fixed positive parameter σ and for any $r \geq 0,\ t \in [0,\ T]$

The following theorems establish a set of properties for the surface power function I. These theorems will be useful in the study of the spatial behaviour of the thermoelastic processes.

Lemma 1: *Let π be a thermoelastic process and $\widehat{D}_T$ be the bounded support of the external data Γ on the time interval $[0,\ T]$. Moreover, let $I(r,\ t)$ be the time-weighted surface power function associated with π and $\mathcal{K}$ be the kinetic energy defined by*

$$\mathcal{K} = \frac{1}{2}(\rho \dot{u}_i \dot{u}_i + \rho\ \chi\ \dot{\varphi}^2). \tag{52}$$

Under the hypotheses of the Section 2, it follows

(I) for $0 \leq r_2 \leq r_1$

$$\begin{aligned} I(r_1,t) \ - \ I(r_2,t) \ &= \ -\int_{B(r_1,r_2)} e^{-\sigma t}[\mathcal{K}(t) + \rho\Psi_M(t)]dv - \\ &-\sigma \int_0^t \int_{B(r_1,r_2)} e^{-\sigma s}\ [\mathcal{K}(s) + \rho\Psi_M(s)]\ \ dv\ ds; \end{aligned} \tag{53}$$

(II) $I(r,\ t)$ is a continuous differentiable function and

$$\begin{aligned} \frac{\partial}{\partial r} I(r,t) \quad &= \quad -\int_{S_r} e^{-\sigma t}\,[\mathcal{K}(t) + \rho\Psi_M(t)]\ \ da - \\ &-\sigma \int_0^t \int_{S_r} e^{-\sigma s}\,[\mathcal{K}(s) + \rho\Psi_M(s)]\ \ dads, \end{aligned} \tag{54}$$

and

$$\frac{\partial}{\partial t} I(r,t) \ = \ -\int_{S_r} e^{-\sigma t}[s_i(t)\dot{u}_i(t) + h(t)\dot{\varphi}(t) - \frac{1}{\theta_0} q(t)\theta(t)]da; \tag{55}$$

(III) for each fixed $t \in [0,\ T]$, $I(r,t)$ is a non–increasing function with respect to r.

Proof. By using the divergence theorem, by eqs. (1-4, 5, 49, 45, 29) and thanks to the definitions of $\widehat{D}_T,\ B_r,\ S_r$, we have for $0 \le r_2 \le r_1$

$$
\begin{aligned}
I(r_1,t)\ -\ I(r_2,t)\ &= -\int_0^t \int_{\partial B(r_1,r_2)} e^{-\sigma s}[s_i(s)\dot{u}_i(s) + h(s)\dot{\varphi}(s) - \frac{1}{\theta_0}\, q(s)\theta(s)]da\, ds = \\
= -\int_0^t \int_{B(r_1,r_2)} & e^{-\sigma s}[S_{ji,j}(s)\dot{u}_i(s)\ + h_{j,j}(s)\dot{\varphi}(s) - \ \frac{1}{\theta_0}\theta(s) q_{i,i}(s)\ +\ S_{ij}(s)\dot{E}_{ij}(s)+ \\
+h_j(s)\dot{\varphi}_{,j}(s) - \ \frac{1}{\theta_0} q_j(s)\theta_{,j}(s)]dv\, ds\ &= -\int_0^t \int_{B(r_1,r_2)} e^{-\sigma s}\frac{\partial}{\partial s}\ [\mathcal{K}(s) + \rho\Psi_M(s)]\ \ dv\, ds.
\end{aligned}
\tag{56}
$$

From (56) we get eq. (53).

The relation (53) implies eq. (54), taking into account the concept of derivative as incremental ratio and performing the limit $r_2 \to r_1$; while, the eq. (55) comes from the definition of $I(r,\ t)$.

The property (III) is obtained from (I) and $\rho,\ \chi$ >0, and by considering that Ψ_M is a norm. ■

Lemma 2: *Under hypotheses of Lemma 1, the surface power function $I(r,\ t)$ associated with π satisfies the following first–order differential inequalities*

$$
\left|\frac{\partial}{\partial t} I(r,t)\ \right| +\ \zeta \frac{\partial}{\partial r} I(r,t)\ \le\ 0, \tag{57}
$$

$$
\frac{\sigma}{\zeta}\ |I(r,t)\ | + \frac{\partial}{\partial r} I(r,\ t)\ \le\ 0, \tag{58}
$$

where

$$
\zeta = \sqrt{\frac{(1+\varepsilon_0)\mu}{\rho_0}}, \tag{59}
$$

and

$$
1+\varepsilon_0 = \frac{1}{2}\ +\ \frac{(K_0+M)}{2\mu}\ \ +\sqrt{\left(-\frac{1}{2}\ +\ \frac{(K_0+M)}{2\mu}\right)^2 + \frac{M}{\mu}}. \tag{60}
$$

Proof. By using the Schwarz's inequality and the arithmetic–geometric mean inequality

and the relations (24, 38) and (47), it follows that

$$
\begin{aligned}
\left| s_i(t)\dot{u}_i(t) + h(t)\dot{\varphi}(t) - \frac{1}{\theta_0} q(t)\theta(t) \right| &\leq \frac{1}{2}\Big[\frac{\varepsilon_1}{\rho_0}\rho\dot{u}_i(t)\dot{u}_i(t) + \frac{1}{\varepsilon_1} S_{ij}(t)S_{ij}(t) + \\
&+ \frac{\varepsilon_1}{\rho_0}\rho\chi\dot{\varphi}^2(t) + \frac{1}{\varepsilon_1}\frac{h_j(t)h_j(t)}{\chi} + \frac{\varepsilon_2}{\rho_0}\frac{\rho c\theta^2(t)}{\theta_0} + \frac{1}{\varepsilon_2}\frac{q_j(t)q_j(t)}{c\,\theta_0} \Big] \leq \\
&\leq \frac{\varepsilon_1}{\rho_0}\mathcal{K}(t) + \frac{1}{2}\Big[\frac{1}{\varepsilon_1}\left((1+\epsilon)2\mu W^*(t) + (1+\frac{1}{\epsilon})M\frac{\rho c\theta^2(t)}{\theta_0}\right) + \\
&+ \frac{\varepsilon_2}{\rho_0}\frac{c\rho\theta^2(t)}{\theta_0} + \frac{1}{\varepsilon_2}K_0\left(2\rho\Psi_M(t) - 2W^*(t) - \frac{\rho c\theta^2(t)}{\theta_0}\right)\Big] \leq \\
&\leq \frac{\varepsilon_1}{\rho_0}\mathcal{K}(t) + \Big[\frac{K_0}{\varepsilon_2}\rho\Psi_M(t) + \left(\frac{1}{\varepsilon_1}(1+\epsilon)\mu - \frac{1}{\varepsilon_2}K_0\right)W^*(t) + \\
&+ \frac{1}{2}\left(\frac{1}{\varepsilon_1}(1+\frac{1}{\epsilon})M + \frac{\varepsilon_2}{\rho_0} - \frac{K_0}{\varepsilon_2}\right)\frac{\rho c\theta^2(t)}{\theta_0}\Big],
\end{aligned}
\tag{61}
$$

for every $\varepsilon_1 > 0,\ \varepsilon_2 > 0,\ \varepsilon > 0$. We choose $\varepsilon,\ \varepsilon_1,\ \varepsilon_2$ such that

$$
\frac{\varepsilon_1}{\rho_0} = \frac{K_0}{\varepsilon_2}, \qquad \frac{1}{\varepsilon_1}(1+\epsilon)\mu - \frac{K_0}{\varepsilon_2} = 0\,, \qquad \frac{1}{\varepsilon_1}(1+\frac{1}{\epsilon})M + \frac{\varepsilon_2}{\rho_0} - \frac{K_0}{\varepsilon_2} = 0;
$$

consequently, we obtain

$$
\varepsilon_1 = \rho_0\zeta, \qquad \varepsilon_2 = \frac{K_0}{\zeta},
$$

where ζ is given by (59) and ε_0 is given by (60). In other words, ε_0 is the positive root of the algebraic equation

$$
\varepsilon^2 - \varepsilon(-1 + \frac{K_0 + M}{\mu}) - \frac{M}{\mu} = 0. \tag{62}
$$

Thus, the relations (51, 55, 61) imply

$$
\left|\frac{\partial}{\partial t}I(r,t)\right| \leq \zeta\int_{S_r} e^{-\sigma t}\left[\mathcal{K}(t) + \rho\Psi_M(t)\right] da, \tag{63}
$$

$$
\sigma\,|I(r,t)| \leq \sigma\,\zeta\int_0^t\int_{S_r} e^{-\sigma s}\left[\mathcal{K}(s) + \rho\Psi_M(s)\right] da ds. \tag{64}
$$

By eqs. (54, 63, 64), the inequalities (57, 58) are reached. ■

Lemma 3: *Under hypotheses of Lemma 1, the surface power function $I(r,\ t)$ associated with π is equal to*

$$
\begin{aligned}
I(r,\ t) = &\int_{B_r} e^{-\sigma t}\left[\mathcal{K}(t) + \rho\Psi_M(t)\right]\ dv + \\
&+\sigma\int_0^t\int_{B_r} e^{-\sigma s}\left[\mathcal{K}(s) + \rho\Psi_M(s)\right]\ dvds \geq 0,
\end{aligned}
\tag{65}
$$

for $r \geq 0, t \in [0,\ T]$.

Proof. If B is a bounded body, then the variable r ranges on $[0,\ L]$, where

$$L = max_{\mathbf{x} \in \bar{B}} \{ min_{\mathbf{y} \in \widehat{D}_T^*} |\mathbf{x} - \mathbf{y}| \} < \infty.$$

From the definition of $\widehat{D}_T$ and $I(r,\ t)$, we have $I(L,\ t) = 0$, so that we obtain eq. (65) with the help of eq. (53).

On the other hand, if B is an unbounded body, then, the variable r ranges on $[0,\ \infty)$. Fixed a pair $(r_0,\ t_0)$ in the plane $(r,\ t)$, such that $r_0 \geq \zeta |t - t_0|$, $t_0 \in [0,\ T]$, the inequality (57) implies that

$$\begin{aligned} \frac{d}{dt}\left[I(r_0 + \zeta(t - t_0), t) \right] &\leq 0; &(66)\\ \frac{d}{dt}\left[I(r_0 - \zeta(t - t_0), t) \right] &\geq 0. &(67) \end{aligned}$$

Therefore, we have

$$\begin{aligned} I(r_0 + \zeta(t - t_0), t) &\leq I(r_0 + \zeta(t' - t_0), t') \qquad \text{with } t \geq t', &(68)\\ I(r_0 - \zeta(t' - t_0), t') &\geq I(r_0 - \zeta(t - t_0), t). \qquad \text{with } t \geq t'. &(69) \end{aligned}$$

For $t = t_0$ and $t' = 0$, the relations (51, 68, 69) imply

$$I(r_0, t_0) \leq I(r_0 - \zeta t_0, 0) = 0, \qquad (70)$$

$$0 = I(r_0 + \zeta t_0, 0) \geq I(r_0, t_0), \qquad (71)$$

and so

$$I(r_0, t_0) = 0. \qquad (72)$$

For $r_0 \to \infty$ and, consequently, for any $t_0 \in [0,\ T]$, eq. (72) becomes

$$I(\infty, t_0) = \lim_{r_0 \to \infty} I(r_0, t_0) = 0.$$

From eq. (53) and for $r_1 \to \infty$ we deduce

$$\begin{aligned} I(r,t) = I(r,t) \ - \ I(\infty,t) \ &= \int_{B_r} e^{-\sigma t} \left[\mathcal{K}(t) + \rho \Psi_M(t) \right] \ dv + \\ +\sigma \int_0^t \int_{B_r} e^{-\sigma s} &\left[\mathcal{K}(s) + \rho \Psi_M(s) \right] \ dv ds. \ \blacksquare \end{aligned} \qquad (73)$$

By the properties of the surface power function $I(r,\ t)$ established in Lemma 1-3, we obtain a complete description of the spatial behaviour of the elastic process in question outside of the support of the data.

Theorem 1: *Under hypotheses of Lemma 1, for each fixed* $t \in [0,\ T]$ *we have the following results:*

1) *Domain of influence*

$$I(r,\ t) = \ 0, \qquad\qquad \textit{for } r \geq \zeta t. \tag{74}$$

2) Spatial decay

$$I(r,\ t)\ \leq\ e^{-\frac{\sigma}{\zeta}r} I(0,\ t), \qquad\qquad \textit{for } r \leq \zeta t. \tag{75}$$

Proof. By putting $r_0 = 0$, $t_0 = 0$ in (66), it follows

$$\frac{dI(\zeta t, t)}{dt} \leq 0,$$

and so

$$I(\zeta t, t) \leq I(0, 0) = 0. \tag{76}$$

From Lemma 1 and Lemma 3, we have

$$0 \leq I(r, t) \leq I(\zeta t, t), \qquad\qquad \forall r \geq \zeta t. \tag{77}$$

Therefore, by the inequalities (65, 76, 77) we obtain the equation (74).

On the other hand, the inequalities (58, 65) imply

$$\frac{\partial}{\partial r}\,[e^{\frac{\sigma}{\zeta}r}\, I(r,t)\,] \leq 0, \qquad\qquad \text{for } r \leq \zeta t, \tag{78}$$

so that we arrive to (75). ■

We remark that the relations (45, 65, 74) give

$$\begin{aligned} 0 \ &= \ I(r,t) = \int_{B_r} e^{-\sigma t}\{\mathcal{K}(t) + [W^*(t) + \frac{\rho c \theta^2(t)}{2\theta_0} - \\ &-\frac{1}{\pi}\int_0^\infty \omega\{\dot{K}^S_{ij}(\omega)\bar{\theta}^{tS}_{,i}(\omega)\bar{\theta}^{tS}_{,j}(\omega) + \dot{K}^S_{ij}(\omega)\bar{\theta}^{tC}_{,i}(\omega)\bar{\theta}^{tC}_{,j}(\omega)\}d\omega]\}dv, \quad \text{for } r \geq \zeta t. \end{aligned}$$

Taking into account that ρ, χ and c are strictly positive functions, that $\tilde{W}$ is a positive definite quadratic form and that $\dot{\mathbf{K}}^S$ is negative definite tensor, we have

$$\dot{u}_i = 0, \quad \dot{\varphi} = 0, \quad \theta = 0 \qquad\qquad \text{on } B_r \times [0, T]. \tag{79}$$

Since the external data are null on B_r, the relation (79) yields

$$u_i = 0, \quad \varphi = 0, \quad \theta = 0 \qquad\qquad \text{on } B\backslash D_r \times [0, T] \qquad \text{for } r \geq \zeta t. \tag{80}$$

Now, we establish the domain of influence of the external given data at time T, according to Gurtin [4]. In fact, we show, by putting $t = T$ and $r = \zeta T$ in relation (80), that on $[0, T]$ the external given data have no effect on points outside of $D_{\zeta T}$.

Theorem 2: *Under hypotheses of Lemma 1, it follows*

$$u_i = 0, \quad \varphi = 0, \quad \theta = 0 \qquad \text{on } B \backslash D_{\zeta T} \times [0, T]. \tag{81}$$

As an immediate consequence of Theorem 1, we show the following uniqueness result valid for a bounded or unbounded body:

Theorem 3: *Under hypotheses of Lemma 1, there exists at most one solution for the boundary-initial-value problem.*

Proof. Thanks to the linearity of the problem, we have only to show that the null data imply null solution. Let $\tilde{\mathbf{U}} = \left\{ \tilde{u}_i, \tilde{\varphi}, \tilde{\theta} \right\}$ be a solution of the problem (1, 6, 49, 48) corresponding to null data. In this case, for each $T \in (0, +\infty)$ we have $\widehat{D}_T = \emptyset$ and $I(r, t) = 0$. Then, we can conclude that

$$\tilde{u}_i = 0, \qquad \tilde{\varphi} = 0, \qquad \tilde{\theta} = 0 \qquad \text{on } B \times [0, +\infty).$$

4 An alternative surface measure

In the above section, we have assumed that the parameter σ is strictly positive. We can avoid this restriction by considering a measure of the type

$$P(r, t) = -\int_0^t \int_{S_r} [s_i(s)\dot{u}_i(s) + h(s)\dot{\varphi}(s) - \frac{1}{\theta_0} q(s)\theta(s)] da ds, \tag{82}$$

for $r \geq 0, \ t \in [0, \ T]$. Following the procedure developed in the previous sections, we can easily prove that $P(r, \ t)$ satisfies

$$\frac{\partial}{\partial r} P(r, t) \ \leq \ 0, \qquad \left| \frac{\partial}{\partial t} P(r, t) \right| + \ \zeta \frac{\partial}{\partial r} P(r, t) \ \leq 0. \tag{83}$$

Moreover, we can deduce that $P(r, \ t)$ is equal to the total energy associated to π on B_r, i.e.

$$P(r, t) = \int_{B_r} \ [\mathcal{K}(t) + \rho \Psi_M(t)] \ dv \geq 0, \tag{84}$$

and

$$P(r, t) \ = 0, \qquad \text{for } r \geq \zeta t. \tag{85}$$

On the basis of these results, we establish the following spatial decay result

Theorem 4: *Let π be a thermoelastic process, $\widehat{D}_T$ be the bounded support of the external data Γ on the time interval $[0,\ T]$ and $P(r,\ t)$ be the surface measure associated with π. Under hypotheses of Section 2, for each fixed $t \in [0,\ T]$ we have*

$$Q(r,t) \ \leq \ \left(1-\frac{r}{\zeta t}\right) Q(0,t), \qquad\qquad \textit{for } r \leq \zeta t, \tag{86}$$

where

$$Q(r,\ t) = \int_0^t \ P(r,\ \alpha)\ d\alpha = \int_0^t \int_{B_r} \ [\mathcal{K}(\alpha) + \rho \Psi_M(\alpha)]\ dv d\alpha. \tag{87}$$

Proof. By (85, 87) we have

$$Q(r',t') = \int_{\frac{r'}{\zeta}}^{t'} \ P(r',\alpha) d\alpha,$$

for any choose $r' > 0$, $t' \in (0,\ T]$ so that $r' \leq \zeta t'$. Now, we consider the following changed variable

$$\alpha = \left(1-\frac{r'}{\zeta t'}\right) s + \frac{r'}{\zeta}; \tag{88}$$

clearly, $\alpha \geq s$. We get

$$Q(r',t') = \left(1-\frac{r'}{\zeta t'}\right) \int_0^{t'} \ P(r', \left(1-\frac{r'}{\zeta t'}\right) s + \frac{r'}{\zeta}) ds. \tag{89}$$

On the other hand, as in the Lemma 3, we can prove that

$$\frac{d}{dt} P(r_0 + \zeta(t-t_0), t) \ \leq 0. \tag{90}$$

Thus, by putting $r_0 = r'$, $t_0 = \alpha$ into (90), we reach the result

$$P(r',\alpha) \ \leq P(r' + \zeta(s-\alpha), s) = P\left(\frac{r's}{t'}, s\right) \ \text{with } \alpha \geq s. \tag{91}$$

The first inequality of (83) implies that $P(r,t)$ is non-increasing function of r, such that

$$P\left(\frac{r's}{t'}, s\right) \leq \ P(0,s) \qquad \text{with } s \in [0,t']. \tag{92}$$

For $r' \leq \zeta t'$ and $t' \in [0,\ T]$, we deduce by the inequality (91, 92) that

$$P\left(r', \left(1-\frac{r'}{\zeta t'}\right) s + \frac{r'}{\zeta}\right) \leq \ P(0,t'). \tag{93}$$

Finally, the relations (87, 89, 93) yield to the inequality (86). ■

However, we have to outline here that the decay rate characteristic to the estimate (86) is lower than the established one in the above section.

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